

7. 当 $x \rightarrow 0$ 时, $x - \sin x \cdot \cos x \cdot \cos 2x \sim ax^n$, 则 _____.

解: $x - \sin x \cos x \cos 2x = x - \frac{1}{2} \sin 2x \cos 2x = x - \frac{1}{4} \sin 4x$
 $= x - \frac{1}{4} \left(4x - \frac{(4x)^3}{3!} + o(x^3) \right) = \frac{16}{6} x^3 + o(x^3)$
 $\therefore a = \frac{8}{3}, n=3.$ (选 B)

8. $f'(x)$ 在 $x=a$ 处连续, $\lim_{x \rightarrow a} \frac{f'(x)}{x-a} = -1$, 则 _____.

解: $\lim_{x \rightarrow a} f'(x) = f'(a)$, 由 $\lim_{x \rightarrow a} \frac{f'(x)}{x-a} = -1 < 0$ 知: $\lim_{x \rightarrow a} f'(x) = 0$

$\therefore f'(a) = 0$, 即 $x=a$ 是 $f(x)$ 的驻点.

由极限的保号性, 得: $x \in \hat{U}(a)$, 有 $\frac{f'(x)}{x-a} < 0$

所以: $x=a$ 是 $f(x)$ 的极大值点.

(选 B)

$x \in (a-\delta, a)$	a	$(a, a+\delta)$
$f'(x)$	+	-
$f(x)$	$\nearrow$	$\searrow$

9. $y = \sqrt{x^2+x} \ln(1+\frac{1}{x})$, $x > 0$, 则 _____.

解: $\lim_{x \rightarrow +\infty} \sqrt{x^2+x} \ln(1+\frac{1}{x}) = \lim_{x \rightarrow +\infty} \frac{1}{x} \sqrt{x^2+x} = 1$ (当 $x \rightarrow +\infty$ 时, $\ln(1+\frac{1}{x}) \sim \frac{1}{x}$)
 $\therefore y=1$ 是曲线的一条水平渐近线.

$\lim_{x \rightarrow 0^+} \sqrt{x^2+x} \ln(1+\frac{1}{x}) = \lim_{x \rightarrow 0^+} \sqrt{x(x+1)} \ln \frac{1+x}{x} = \lim_{x \rightarrow 0^+} \sqrt{x+1} \cdot \sqrt{x} (\ln(1+x) - \ln x)$
 $= \lim_{x \rightarrow 0^+} (\sqrt{x} \ln(1+x) - \sqrt{x} \ln x) = 0 - \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-\frac{1}{2}}} = -\lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-\frac{3}{2}}} = 0$

$\therefore$ 曲线无铅直渐近线.

(选 A)

10. 在 $(-\infty, +\infty)$ 内, 方程 $|x|^{\frac{1}{4}} + |x|^{\frac{1}{2}} - \cos x = 0$

解: $f(x) = |x|^{\frac{1}{4}} + |x|^{\frac{1}{2}} - \cos x$, $f(-x) = f(x) \therefore f(x)$ 为偶函数.

1° 当 $x > 0$ 时, $f(0) = -1$, $f(x) = x^{\frac{1}{4}} + x^{\frac{1}{2}} - \cos x$, $f(1) = 2 - \cos 1 < 0$

2° 由 $f'(x) = \frac{1}{4}x^{-\frac{3}{4}} + \frac{1}{2}x^{-\frac{1}{2}} + \sin x$ 由 $\sin x \geq -1$ 知, 在 $(0, 1)$ 内以方程

当 $0 < x < \pi$ 时, $f'(x) > 0$, $f(x)$ 少存在一个根. $f(\pi) = 0$.

$\therefore$ 方程 $f(x) = 0$ 在 $(0, \pi)$ 内只有唯一根.

3° 当 $x > \pi$ 时, $f(x) = |x|^{\frac{1}{4}} + |x|^{\frac{1}{2}} - \cos x > 0$. $\therefore$ 此方程在 $(0, +\infty)$ 内只有一个根

$\therefore$ 由 $f(x)$ 为偶函数, 得此方程 $f(x) = 0$ 在 $(-\infty, +\infty)$ 内有两个根.

(选 C)

二, 填空:

No.4

1. $f(x) = \frac{x^3+x}{(x^2-1)\arctan x} 2^{\frac{1}{x-3}}$ 在可去间断点的个数为 1

解: $x = \pm 1$, ~~$x = \frac{1}{2}$~~ , $x = 0$, $x = 3$ 都是 $f(x)$ 的间断点.

$$\lim_{x \rightarrow 1} \frac{x^3+x}{(x^2-1)\arctan x} 2^{\frac{1}{x-3}} = \infty,$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{x^3+x}{(x^2-1)\arctan x} \cdot 2^{\frac{1}{x-3}} = \lim_{x \rightarrow -1} \frac{x(x^2+1) \cdot 2^{\frac{1}{x-3}}}{(x+1)(x-1)\arctan x} = \infty$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{x^3+x}{(x^2-1)\arctan x} 2^{\frac{1}{x-3}} = \lim_{x \rightarrow 0} \frac{x(x^2+1)}{(x^2-1) \cdot x} 2^{\frac{1}{x-3}} = -2^{-\frac{1}{3}}$$

~~$\lim_{x \rightarrow \frac{1}{2}} f(x) = \lim_{x \rightarrow \frac{1}{2}} \frac{x^3+x}{(x^2-1)\arctan x} 2^{\frac{1}{x-3}}$~~

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{x^3+x}{(x^2-1)\arctan x} 2^{\frac{1}{x-3}} = \infty$$

$\therefore x = 0$ 是 $f(x)$ 的可去间断点.

2. $\lim_{x \rightarrow \infty} f(x)$ 存在, $\lim_{x \rightarrow \infty} 3xf(x) = \lim_{x \rightarrow \infty} [4f(x) + 5]$, 则 $\lim_{x \rightarrow \infty} xf(x) = \underline{\frac{5}{3}}$.

解: $\exists \lim_{x \rightarrow \infty} f(x) = a \quad \therefore \lim_{x \rightarrow \infty} 3xf(x) = 4a + 5$

$\frac{1}{3} \lim_{x \rightarrow \infty} 3xf(x)$ 存在, 由 $\lim_{x \rightarrow \infty} 3xf(x)$ 存在 $\therefore \lim_{x \rightarrow \infty} f(x) = 0 = a$

$$\therefore \lim_{x \rightarrow \infty} xf(x) = \frac{5}{3}$$

3. $\lim_{x \rightarrow 0} (e^x + e^{-x} - 1) \ln(1 + \pi x^2)$ " $\frac{0}{0}$ "

$$= \lim_{x \rightarrow 0} \left\{ [1 + (e^x + e^{-x} - 2)] e^{\frac{1}{e^x + e^{-x} - 2}} \right\} \frac{e^x + e^{-x} - 2}{\ln(1 + \pi x^2)}$$

$$\approx \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{\ln(1 + \pi x^2)} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{\pi x^2} = \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{2\pi x}$$

$$= \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{2\pi} = \frac{1}{\pi}$$

$\therefore$ 原式 $= e^{\frac{1}{\pi}}$.

4. $\begin{cases} x = \ln(1+\theta^2) \\ y = \theta - \arctan \theta \end{cases}$, 则 $\frac{d^2y}{dx^2} \Big|_{\theta=1} = \frac{1}{2}$.

解: $\frac{dy}{dx} = \frac{y'_\theta}{x'_\theta} = \frac{1 - \frac{1}{1+\theta^2}}{\frac{2\theta}{1+\theta^2}} = \frac{\theta}{2}$

$\frac{d^2y}{dx^2} = \frac{(y')'_\theta}{x'_\theta} = \frac{\frac{1}{2}}{\frac{2\theta}{1+\theta^2}} \therefore \frac{d^2y}{dx^2} \Big|_{\theta=1} = \frac{1}{2}$

5. $y - 2e^y \sin x + x = 0$, 则 $y''(0) = 4$.

解: $y' - 2e^y \cdot y' \sin x - 2e^y \cos x + 1 = 0$ ①

$y'' - 2e^y (y')^2 \sin x - 2e^y y'' \sin x - 2e^y y' \cos x - 2e^y y' \cos x + 2e^y \sin x = 0$ ②

当 $x=0$ 时, $y=0$. 代入①得: $y'(0) = 1$.

代入②得: $y''(0) = 4$

6. $y = \frac{1}{x^2+3x+2}$, 则 $y^{(n)} =$ _____.

解: $y = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

$y^{(n)} = \left(\frac{1}{x+1}\right)^{(n)} - \left(\frac{1}{x+2}\right)^{(n)} = (-1)^n \frac{n!}{(x+1)^{n+1}} - (-1)^n \frac{n!}{(x+2)^{n+1}}$

7. $y = \sqrt{\frac{1+x}{1-x}} \cdot \sqrt[4]{\frac{1+2x}{1-2x}} \cdot \sqrt[6]{\frac{1+3x}{1-3x}}$, 则 $dy \Big|_{x=0} =$ _____.

解: $\ln y = \frac{1}{2}(\ln(1+x) - \ln(1-x)) + \frac{1}{4}(\ln(1+2x) - \ln(1-2x)) + \frac{1}{6}(\ln(1+3x) - \ln(1-3x))$

$\therefore \frac{1}{y} y' = \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) + \frac{1}{4} \left(\frac{2}{1+2x} + \frac{2}{1-2x} \right) + \frac{1}{6} \left(\frac{3}{1+3x} + \frac{3}{1-3x} \right)$

当 $x=0$ 时, $y=1$. $\therefore y' \Big|_{x=0} = 3$. $\therefore dy \Big|_{x=0} = 3dx$.

8. 已知 $x > -1$, 根据拉格朗日中值定理, $\exists \theta \in (0,1)$, 使 $\ln(1+x) = \frac{x}{1+\theta x}$.

解: $\theta = \frac{1}{\ln(1+x)} - \frac{1}{x}$ 则 $\lim_{x \rightarrow 0} \theta = \frac{1}{2}$.

$\lim_{x \rightarrow 0} \theta = \lim_{x \rightarrow 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x \ln(1+x)}$

$= \lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x}}{2x} = \lim_{x \rightarrow 0} \frac{1}{2(1+x)} = \frac{1}{2}$.

9. $f(x) = \frac{(e^x - e^{-x})^2}{e^x + e^{-x}} + x^2 \cos 2023x$ 第皮亚诺余项的 n 阶麦克劳林

No.6.

的展开式为 $f(x) = a_0 + a_1x + \dots + a_nx^n + o(x^n)$

则在上面的展开式中 x^{2023} 的系数 $a_{2023} = \underline{0}$.

解: $f(-x) = f(x)$, $\therefore f(x)$ 是偶函数

$\therefore f'(x)$ 是奇函数, $f''(x)$ 是偶函数

$\therefore f^{(2023)}(x)$ 是奇函数, $f^{(2023)}(0) = 0$.

$$a_{2023} = \frac{f^{(2023)}(0)}{2023!} = 0$$

10. $y = x^3 - 6x^2 + 9x + 6$ 的拐点 _____.

解: $y' = 3x^2 - 12x + 9$, $y'' = 6x - 12 = 6(x - 2)$

令 $y'' = 0$, 得 $x = 2$.

x	$(-\infty, 2)$	2	$(2, +\infty)$
y''	-	0	+
y		拐点	

$\therefore$ 拐点 $(2, 8)$